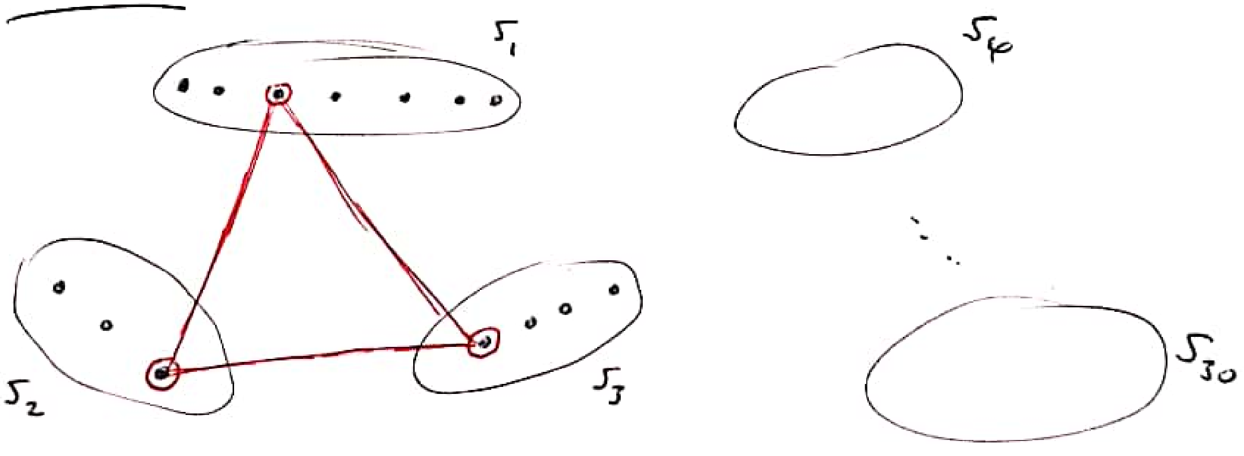
PROBLEM. 1989 POINTS ARE GIVEN IN THE PLANE, NO THREE OF WHICH ARE COLLINEAR. DIVIDE THESE POINTS INTO 30 GROUPS WITH A DIFFERENT NUMBER OF POINTS IN EACH GROUP.

LET N BE THE NUMBER OF TRIANGLES WITH VERTICES IN DIFFERENT GROUPS.

HOW SHOULD THE POINTS BE DIVIDED IN ORDER TO MAXIMIZE N?

SOLUTION



Suppose we divide into 30 groups $S_1, S_2, \dots, S_{30}$ with x_i points in group S_i $(i = 1, 2, \dots, 30)$

THEREFORE $x_i \geq 1$ FOR ALL i, WE HAVE $x_i \neq x_j$ FOR ALL i, j, AND

$$\sum_{i=1}^{30} x_i = 1989.$$

THE NO. OF TRIANGLES WITH VERTICES IN GROUPS

S_i, S_j, S_k IS $x_i x_j x_k$.

THEREFORE

$$N = x_1 x_2 x_3 + x_1 x_2 x_4 + x_2 x_3 x_4 + \cdots + x_{28} x_{29} x_{30}$$

$$= \sum_{i<j<k} x_i x_j x_k$$

NOW CONSIDER FIXING $x_3, x_4, \ldots, x_{30}$ AND ARRANGING (x_1, x_2) SUBJECT TO THE CONSTRAINT $x_1 + x_2 = \text{CONSTANT}$.

SUPPOSE WE MOVE POINTS FROM S_2 INTO S_1, WHERE $x_1 \geq x_2$.

THEN FOR POSITIVE INTEGER k, WE GET

$$x_1 \longmapsto x_1 + k$$
$$x_2 \longmapsto x_2 - k$$

THE PRODUCT

$$x_1 x_2 \longmapsto (x_1+k)(x_2-k) = x_1 x_2 - k(x_1 - x_2) - k^2$$
$$< x_1 x_2$$

since $k>0$, $x_1 > x_2$, and $k^2 > 0$.

i.e., THE PRODUCT $x_1 x_2$ DECREASES.

So

$$N = x_1 x_2 (x_3 + x_4 + \cdots + x_{30})$$

$$+ (x_1 + x_2)(x_3 x_4 + x_4 x_5 + x_3 x_5 + \cdots + x_{29} x_{30})$$

$$+ (x_3 x_4 x_5 + x_3 x_5 x_6 + \cdots + x_{28} x_{29} x_{30})$$

$$= x_1 x_2 F_1 + (x_1 + x_2) F_2 + F_3$$

WHERE F_1, F_2, F_3 ARE POSITIVE & DO NOT DEPEND ON x_1 AND x_2.

NOW, MOVING x_1 AND x_2 FURTHER APART MUST DECREASE N

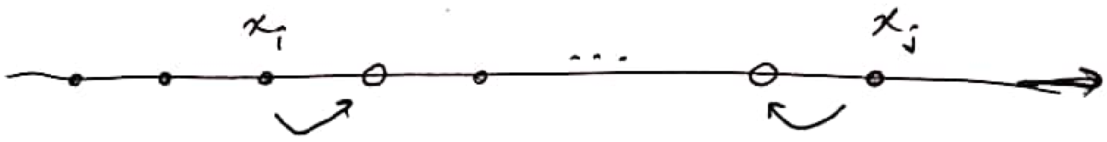
[$x_1 > x_2$, $\quad x_1 \mapsto x_1 + h$, $x_2 \mapsto x_2 - h$ $\quad (h > 0)$

DECREASES THE PRODUCT $x_1 x_2$ AND THEREFORE DECREASES N]

SO MOVING ANY 2 x_i'S CLOSER TOGETHER MUST INCREASE N.

FIRST WE NOTE THAT SINCE THERE IS A FINITE NUMBER OF ARRANGEMENTS FOR THE x_i'S, A MAXIMUM EXISTS FOR N.

ALSO, USING THE ARGUMENT ABOVE, WE SEE THAT A MAXIMAL ARRANGEMENT OF x_i'S ON THE NUMBER LINE CANNOT HAVE 2 OR MORE "GAPS", i.e.,



AS THIS ARRANGEMENT WOULD MEAN THAT YOU COULD MOVE x_i & x_j CLOSER TOGETHER, THUS INCREASING N, WHICH CONTRADICTS THE ASSUMPTION THAT N IS MAXIMAL.

SO THE MAXIMAL ARRANGEMENT CANNOT HAVE MORE THAN ONE "GAP", i.e., IT MUST BE A SEQUENCE OF CONSECUTIVE INTEGERS WITH ONE REMOVED (POSSIBLY AT THE BEGINNING).

SO WE NEED TO FIND a, k (INTEGERS) SUCH THAT

$$a + (a+1) + (a+2) + \cdots + (a+30) - (a-k) = 1989$$

HERE $k \in \{0, 1, \ldots 29\}$

THIS GIVES

$$31a + (1+2+\cdots+30) = 1989 + (a+k)$$

$$30a + \frac{30(31)}{2} = 1989 + k$$

$$\underline{30a - k = 1524}$$

$\Rightarrow k \equiv 6 \pmod{30}$ AND SO $\underline{k = 6}$.

SO $30a = 1530$

$\Rightarrow \underline{a = 51}$.

SO WE TAKE THE CONSECUTIVE INTEGERS $\{51, 52, \ldots, 81\}$ AND LEAVE OUT $a+k = 57$.

<u>ANSWER</u>: THE NUMBERS OF POINTS IN THE GROUPS ARE $\{51, 52, 53, 54, 55, 56, 58, 59, \ldots, 81\}$.

<u>QED</u>.